

CONVEX COMPACT SETS THAT ADMIT A LOWER SEMICONTINUOUS STRICTLY CONVEX FUNCTION

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ABSTRACT. We study the class of compact convex subsets of a topological vector space which admits a strictly convex and lower semicontinuous function. We prove that such a compact set is embeddable in a strictly convex dual Banach space endowed with its weak* topology. In addition, we find exposed points where a strictly convex lower semicontinuous function is continuous.

1. INTRODUCTION

A well-known result of Hervé [6] says that a compact convex subset $K \subset X$ of a locally convex space is metrizable if and only if there exists $f: K \rightarrow \mathbb{R}$ which is both continuous and strictly convex. It happens that lower semicontinuity is a very natural hypothesis for a convex function, so it is natural to wonder if the existence of a strictly convex lower semicontinuous function on compact convex subset $K \subset X$ of a locally convex space enforces special topological properties on K . Ribarska proved [16, 17] that such a compact is *fragmentable* by a finer metric, and in particular it contains a completely metrizable dense subset. The third named author proved [14] that the same is true for the set of its extreme points $\text{ext}(K)$. On the other hand, Talagrand's argument in [2, Theorem 5.2.(ii)] shows that $[0, \omega_1]$ is not embeddable in such a compact set. In addition, Godefroy and Li showed [5] that if the set of probabilities on a compact group K admits a strictly convex lower semicontinuous function then K is metrizable.

Our purpose here is to continue with the study of the class of compact convex subsets which admits a strictly convex lower semicontinuous function. We shall denote this class by \mathcal{SC} . The first remarkable fact that we have got is a Banach representation result.

Theorem 1.1. *Let X be a locally convex topological vector space and let $K \subset X$ be convex compact subset. Then there exists a function $f: K \rightarrow \mathbb{R}$ which is both lower semicontinuous and strictly convex if and only if K imbeds linearly into a strictly convex dual Banach space Z endowed with its weak* topology.*

Notice that the strictly convex norm of the dual Banach space in the statement is weak* lower semicontinuous, which is a stronger condition than just being a strictly convex Banach space isomorphic to a dual space.

If $f: K \rightarrow \mathbb{R}$ is a strictly convex function, then the *symmetric* defined by

$$\rho(x, y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)$$

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provides a consistent way to measure *diameters* of subsets of K . This idea was successfully applied in renorming theory [9]. We will prove that every nonempty subset of K has slices of arbitrarily small ρ -diameter, we can mimic some arguments of the geometric study of the Radon–Nikodým property which leads to results as the following one.

Theorem 1.2. *Let X be a locally convex topological vector space and let $f: X \rightarrow \mathbb{R}$ be lower semicontinuous, strictly convex and bounded on compact sets. Then for every $K \subset X$ compact and convex, the set of points in K which are both exposed and continuity points of $f|_K$ is dense in $\text{ext}(K)$.*

The organization of the paper is as follows. In the second section we present stability properties of the class \mathcal{SC} , which allow us to prove the embedding Theorem 1.1. The third and fourth sections are devoted to the search of faces and exposed points of continuity, respectively. Finally, a characterization of the class \mathcal{SC} in terms of the existence of a symmetric with countable dentability index is given in Section 5.

2. EMBEDDING INTO A DUAL SPACE

Along this section X will denote a locally convex topological vector space. Our first goal is to study the properties of following class of compact sets.

Definition 2.1. The class $\mathcal{SC}(X)$ consists of all the nonempty compact convex subsets K of X such that there exists a function $f: K \rightarrow \mathbb{R}$ which is lower semicontinuous and strictly convex. In addition, \mathcal{SC} denotes the class composed of all the families $\mathcal{SC}(X)$ for any locally convex space X .

Since a lower semicontinuous function on a compact space attains its minimum, the function f is bounded below. Later we shall show that we may always take f to be bounded. Notice that metrizable convex compacts admits continuous strictly convex functions, so they are in the class. In particular, if X is metrizable then $\mathcal{SC}(X)$ contains all the convex compact subsets of X . If X is a Banach space endowed with its weak topology, then $\mathcal{SC}(X)$ is made up of all convex weakly compact subsets as a consequence of the strictly convex renorming results for WCG spaces.

Proposition 2.2. *The class \mathcal{SC} satisfies the following stability properties:*

- a) $\mathcal{SC}(X)$ is stable by translations and homothetics;
- b) \mathcal{SC} is stable by Cartesian products;
- c) \mathcal{SC} is stable by linear continuous images;
- d) If $A, B \in \mathcal{SC}(X)$, then $A + B \in \mathcal{SC}(X)$.

Proof. Statement a) is obvious. To prove b) suppose that f_i witnesses $A_i \in \mathcal{SC}(X_i)$ for $i = 1, \dots, n$. Then $\sum_{i=1}^n f_i \circ \pi_i$, where $\pi_i: \bigotimes_{i=1}^n X_i \rightarrow X_i$ is the coordinate projection, witnesses that $A_1 \times \dots \times A_n \in \mathcal{SC}(\bigotimes_{i=1}^n X_i)$.

To prove c) assume that $A \in \mathcal{SC}(X)$ and $T: X \rightarrow Y$ is linear and continuous. Obviously $T(A)$ is convex and compact. Let $f: A \rightarrow \mathbb{R}$ be lower semicontinuous and strictly convex. It is straightforward to check that the function $g: T(A) \rightarrow \mathbb{R}$ defined by

$$g(y) = \inf \{f(x) : x \in T^{-1}(y)\}$$

does the work. Finally, d) follows by a combination of b) and c). \square

We will need a kind of external convex sum of convex compact sets.

Definition 2.3. Given $A, B \subset X$ convex compact define a subset of $X \times X \times \mathbb{R}$ by

$$A \oplus B = \{(\lambda x, (1 - \lambda)y, \lambda) : x \in A, y \in B, \lambda \in [0, 1]\}.$$

Lemma 2.4. *Let $A, B \subset X$ be convex compact subsets. Then*

- a) $A \oplus B$ is a convex compact subset of $X \times X \times \mathbb{R}$;
- b) if $f: A \rightarrow \mathbb{R}$ and $g: B \rightarrow \mathbb{R}$ are convex, then $h: A \oplus B \rightarrow \mathbb{R}$ defined by

$$h((\lambda x, (1 - \lambda)y, \lambda)) = \lambda f(x) + (1 - \lambda)g(y)$$

is convex as well;

- c) if $A, B \in \mathcal{SC}(X)$, then $A \oplus B \in \mathcal{SC}(X \times X \times \mathbb{R})$.

Proof. Compactness is clear in statement a). Given $(\lambda_i x_i, (1 - \lambda_i)y_i, \lambda_i) \in A \oplus B$ for $i = 1, 2$, just observe that

$$\begin{aligned} & \left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1 - \lambda_1)y_1 + (1 - \lambda_2)y_2}{2}, \frac{\lambda_1 + \lambda_2}{2} \right) \\ &= \left(\frac{\lambda_1 + \lambda_2}{2} \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}, \left(1 - \frac{\lambda_1 + \lambda_2}{2}\right) \frac{(1 - \lambda_1)y_1 + (1 - \lambda_2)y_2}{(1 - \lambda_1) + (1 - \lambda_2)}, \frac{\lambda_1 + \lambda_2}{2} \right) \end{aligned}$$

(the case where $\lambda_1 = \lambda_2 = 0, 1$ can be handed in a different way). Thus, $A \oplus B$ is convex. For the convexity of function h notice that

$$\begin{aligned} & h\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1 - \lambda_1)y_1 + (1 - \lambda_2)y_2}{2}, \frac{\lambda_1 + \lambda_2}{2}\right) \\ &= \frac{\lambda_1 + \lambda_2}{2} f\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}\right) + \left(1 - \frac{\lambda_1 + \lambda_2}{2}\right) g\left(\frac{(1 - \lambda_1)y_1 + (1 - \lambda_2)y_2}{(1 - \lambda_1) + (1 - \lambda_2)}\right) \\ &\leq \frac{\lambda_1 + \lambda_2}{2} \frac{\lambda_1 f(x_1) + \lambda_2 f(x_2)}{\lambda_1 + \lambda_2} + \left(1 - \frac{\lambda_1 + \lambda_2}{2}\right) \frac{(1 - \lambda_1)g(y_1) + (1 - \lambda_2)g(y_2)}{(1 - \lambda_1) + (1 - \lambda_2)} \\ &= \frac{1}{2} (h((\lambda_1 x_1, (1 - \lambda_1)y_1, \lambda_1)) + h((\lambda_2 x_2, (1 - \lambda_2)y_2, \lambda_2))) . \end{aligned}$$

If f and g were strictly convex, the above inequality for h would become strict if $x_1 \neq x_2$ or $y_1 \neq y_2$. To overcome this difficulty consider the function

$$k((\lambda x, (1 - \lambda)y, \lambda)) = h((\lambda x, (1 - \lambda)y, \lambda)) + \lambda^2$$

and notice that λ^2 provides the strict inequality when $x_1 = x_2$ and $y_1 = y_2$. \square

Proposition 2.5. *Suppose that $A, B \in \mathcal{SC}(X)$. Then $\text{conv}(A \cup B) \in \mathcal{SC}(X)$ and $\text{aconv}(A) \in \mathcal{SC}(X)$.*

Proof. Consider the map $T: X \times X \times \mathbb{R} \rightarrow X$ defined by $T((x, y, t)) = x + y$ and observe that $T(A \oplus B) = \text{conv}(A \cup B)$. Since T is linear and continuous, the combination of the previous results gives us that $\text{conv}(A \cup B) \in \mathcal{SC}(X)$. The application to the symmetric convex hull follows by applying it with $B = -A$. \square

Lemma 2.6. *Let $B \subset X$ be a symmetric compact convex set and let $Z = \text{span}(B)$. Then the following hold:*

- a) Z , with the norm given by the Minkowski functional of B , is isometric to a dual Banach space;
- b) B imbeds linearly into (Z, w^*) ;
- c) if $f: X \rightarrow \mathbb{R}$ is convex and lower semicontinuous, then $f|_Z$ is weak* lower semicontinuous.

Proof. Notice that $Z = \bigcup_{n=1}^{\infty} nB$, and thus the Minkowski functional of B is a norm on Z . Of course, B is the unit ball of Z endowed with this norm. By a result of Dixmier-Ng, see for instance [10], the space Z is isometric to the dual of the Banach space W of all linear functionals f on Z such that $f|_B$ is τ -continuous. If $f: X \rightarrow \mathbb{R}$ is convex and lower semicontinuous, then the sets $\{f \leq a\}$ are convex and closed for any $a \in \mathbb{R}$. We have $\{f|_Z \leq a\} = \{f \leq a\} \cap Z$, and thus $\{f|_Z \leq a\} \cap nB = \{f \leq a\} \cap nB$ is compact, and so it is weak* compact as subset

of Z for every $n \in \mathbb{N}$. By the Banach-Dieudonné theorem, $\{f|_Z \leq a\}$ is a weak* closed subset of Z . \square

Proof of Theorem 1.1. Let $B = \text{aconv}(K)$ which is in $\mathcal{SC}(X)$. The function f witnessing that $B \in \mathcal{SC}(X)$ is weak* lower semicontinuous and strictly convex. By Lemma 2.6 we only need to renorm the dual space Z . Notice that the function f can be taken symmetric and bounded. Indeed, for the symmetry just take $g(x) = f(x) + f(-x)$. Now apply the Baire theorem to the $B = \bigcup_{n=1}^{\infty} g^{-1}((-\infty, n])$ to obtain a set of the form λB with $\lambda > 0$ where g is bounded. Then redefine f as $f(x) = g(\lambda x)$.

Without loss of generality we may assume that f takes values in $[0, 1]$. Consider the function defined on B_Z by

$$h(x) = \frac{1}{2}(3\|x\| + f(x))$$

and consider the set $C = \{x \in B_Z : h(x) \leq 1\}$. Clearly $\frac{1}{3}B_Z \subset C \subset \frac{2}{3}B_Z$, and C is convex, symmetric and weak* closed. Moreover, if $h(x) = h(y) = 1$, then $h(\frac{x+y}{2}) < 1$. Therefore, C is the unit ball of an equivalent strictly convex dual norm on Z . \square

Corollary 2.7. *If $K \in \mathcal{SC}(X)$, then it is witnessed by the square of a lower semicontinuous strictly convex norm defined on $\text{span}(K)$.*

We shall finish this section by showing the connection between the class \mathcal{SC} and $(*)$ property. The following notion was introduced in [11] in order to characterize dual Banach spaces that admit a dual strictly convex norm:

Definition 2.8. A compact space K is said to have $(*)$ if there exists a sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ of families of open subsets of K such that, given any $x, y \in K$, there exists $n \in \mathbb{N}$ such that:

- a) $\{x, y\} \cap \bigcup \mathcal{U}_n$ is non-empty;
- b) $\{x, y\} \cap U$ is at most a singleton for every $U \in \mathcal{U}_n$.

Here we are using the agreement that $\bigcup \mathcal{U}_n = \bigcup \{U : U \in \mathcal{U}_n\}$. Recall that if K is a subset of a locally convex topological vector space then a slice of K is an intersection of K with an open halfspace. If the elements of $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ can be taken to be slices of K , then K is said to have $(*)$ *with slices*. It is shown in [11, Theorem 2.7] that if Z is a dual Banach space then (B_Z, w^*) has $(*)$ with slices if and only if Z admits a dual strictly convex norm.

Corollary 2.9. *Let (X, τ) be locally convex topological vector space and $K \subset X$ be compact and convex. Then $K \in \mathcal{SC}(X)$ if and only if K has $(*)$ with slices.*

Proof. By Lemma 2.6 we may assume that $K \subset Z = \text{span}(K)$ has $(*)$ with weak* slices. It follows from [11, Proposition 2.2] that then there is a lower semicontinuous strictly convex function defined on K . On the other hand, assume that ϕ witnesses $K \in \mathcal{SC}(X)$. For $f \in (X, \tau)^*$ and $r \in \mathbb{R}$, denote $S(f, r) = \{x \in K : f(x) > r\}$. Consider the families $\{\mathcal{U}_{qr}\}_{q, r \in \mathbb{Q}}$ of open subsets given by

$$\mathcal{U}_{qr} = \{S(f, r) : f \in (X, \tau)^*, S(f, r) \cap \{x : \phi(x) \leq q\} = \emptyset\}.$$

Let $x \neq y$ be in K . We may assume that $\phi(x) \leq \phi(y)$. Since ϕ is strictly convex, there exists $q \in \mathbb{Q}$ such that $\phi(\frac{x+y}{2}) < q < \phi(y)$. By the Hahn-Banach theorem, there is $f \in (X, \tau)^*$ and $r \in \mathbb{Q}$ such that $\sup\{f(z) : \phi(z) \leq q\} < r < f(y)$. Therefore, $S(f, r) \cap \{z : \phi(z) \leq q\} = \emptyset$ and $\{x, y\} \cap \bigcup \mathcal{U}_{qr} \neq \emptyset$.

Suppose that $x, y \in S(g, r) \in \mathcal{U}_{qr}$. Then $g(x), g(y) > r$ implies $g(\frac{x+y}{2}) > r$. Hence $\frac{x+y}{2} \notin \{z : \phi(z) \leq q\}$, a contradiction. So $\{x, y\} \cap S(g, q)$ is at most a singleton for each $S(g, q) \in \mathcal{U}_{qr}$. \square

3. FACES OF CONTINUITY

We will assume along the section that $Z = W^*$ is a dual Banach space endowed with the weak* topology. Therefore any unspecified topological concept (compact, open, ...) is always referred to the weak* topology. The elements of W will be considered as functionals on Z . Other topological ingredient that we will use is a symmetric $\rho: Z \times Z \rightarrow [0, +\infty)$. Recall that a symmetric satisfies $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) = 0$ if and only if $x = y$. Since a symmetric does not satisfy the triangle inequality, its associated topology is complicated to handle. Nevertheless we have a natural notion of diameter associated to ρ defined by

$$\rho\text{-diam}(A) = \sup\{\rho(x, y) : x, y \in A\}$$

Let us recall the definition of face of a convex set.

Definition 3.1. Let $C \subset Z$ be closed and convex. We say that a closed subset $F \subset C$ is a face if there is a continuous affine function $w: C \rightarrow \mathbb{R}$ such that

$$F = \{x \in C : w(x) = \sup\{w, C\}\}.$$

In that case we say that the face is produced by w . In addition, we say that a point $x \in C$ is a exposed point of C if $\{x\}$ is a face of C .

Sometimes the face is produced by an element of the dual. Nevertheless, there may exist continuous affine functions on C that are not the restriction of an element of the dual.

We shall need the following lemma.

Lemma 3.2 (Lemma 3.3.3 of [1]). *Suppose that $w \in W$ and $\|w\| = 1$. For $r > 0$ denote by V_r the set $rB_Z \cap w^{-1}(0)$. Assume that x_0 and y are points of Z such that $w(x_0) > w(y)$ and $\|x_0 - y\| \leq r/2$. If $u \in W$ satisfies that $\|u\| = 1$ and $u(x_0) > \sup\{u, y + V_r\}$, then $\|w - u\| \leq \frac{2}{r}\|x_0 - y\|$.*

First we shall discuss the dual Banach case.

Proposition 3.3. *Let $f: Z \rightarrow \mathbb{R}$ be a convex lower semicontinuous function which is bounded on compact subsets. If $K \subset Z$ is compact convex, then there exists a G_δ dense set of elements of W producing faces where $f|_K$ is constant and continuous.*

Proof. Define the pseudo-symmetric ρ by the formula

$$\rho(x, y) = \frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2.$$

We claim that $\rho(x, y) = 0$ implies $f(x) = f(y) = f(\frac{x+y}{2})$ (in particular, if f were strictly convex, ρ would be a symmetric). Indeed, it follows easily from this observation

$$\rho(x, y) \geq \frac{f(x)^2 + f(y)^2}{2} - \left(\frac{f(x) + f(y)}{2}\right)^2 = \left(\frac{f(x) - f(y)}{2}\right)^2 \geq 0.$$

Now we claim that the set $G(K, \varepsilon)$ is open and dense in W for $K \subset Z$ compact convex and $\varepsilon > 0$, where

$$G(K, \varepsilon) = \{w \in W : \exists a < \sup\{w, K\}, \rho\text{-diam}(K \cap \{w > a\}) < \varepsilon\}.$$

Suppose that $w \in G(K, \varepsilon)$. If $w' \in W$ is close enough to w to fulfill that

$$\sup\{w', K\} > \sup\{w', K \cap \{w \leq a\}\}$$

then $w' \in G(K, \varepsilon)$ as well. Thus $G(K, \varepsilon)$ is open. In order to see that it is also dense, fix $w \in W$ and $\delta \leq 1/4$. Take $x \in K$ and $y \in Z$ with $w(x) > a > w(y)$ for some $a \in \mathbb{R}$. Take $r = \sup\{\|x' - y\|, x' \in K\}/2\delta$, consider the set V_r given by Lemma 3.2 and define the set $C = \text{conv}(K \cup (y + V_r))$. By [14, Theorem 1.1], the

halfspace $\{w > a\}$ contains a point $x_0 \in \text{ext}(C)$ where $f|_C$ is continuous. Notice that $x_0 \in \text{ext}(K)$ and $\|x_0 - y\| \leq r/2$. There exists $u \in W$ and $b \in \mathbb{R}$ such that $u(x_0) > b$, $C \cap \{u > b\} \subset C \cap \{w > a\}$ and $\rho\text{-diam}(C \cap \{u > b\}) < \varepsilon$. In particular $\rho\text{-diam}(K \cap \{u > b\}) < \varepsilon$. Since $C \cap \{u > b\}$ does not meet $y + V_r$, we have $u(x_0) > \sup\{u, y + V_r\}$. Thus, $\|w - u\| \leq \frac{2}{r}\|x_0 - y\| \leq \delta$. That completes the proof of the density of $G(K, \varepsilon)$ in W .

By the Baire theorem, the set $G(K) = \bigcap_{n=1}^{\infty} G(K, 1/n)$ is dense. If $w \in G(K)$ and $s = \sup\{w, K\}$ then

$$\lim_{t \rightarrow s^-} \rho\text{-diam}(K \cap \{w > t\}) = 0.$$

In particular, the face $F = K \cap \{w = s\}$ satisfies that $\rho\text{-diam}(F) = 0$. That implies that f is constant on F . Moreover, we claim that any point $x \in F$ is a point of continuity of $f|_K$. If $(x_\alpha) \subset K$ is a net with limit x , then $\lim_\alpha w(x_\alpha) = w(x)$. Therefore $\lim_\alpha \rho(x_\alpha, x) = 0$. It follows that $\lim_\alpha f(x_\alpha) = f(x)$, so $f|_K$ is continuous at x . \square

Now the above result can be translated into a more general setting.

Proposition 3.4. *Let $f: X \rightarrow \mathbb{R}$ be a convex lower semicontinuous function which is bounded on compact subsets. Then for every compact convex subset $K \subset X$ and every open slice $S \subset K$, there is a face $F \subset S$ of K such that $f|_K$ is constant and continuous on F .*

Proof. By Lemma 2.6, $Z = \bigcup_{n=1}^{\infty} n\text{aconv}(K)$ is a dual Banach space and $f|_Z$ is weak* lower semicontinuous. Then we can apply the previous proposition. \square

It is clear that the last two results are true for countably many functions simultaneously.

Remark 3.5. We do not know if the function f in Proposition 3.3 and 3.4 can be assumed to be defined only on K . Notice that if $\|\cdot\|$ is a strictly convex norm on Z then $f(x) = -\sqrt{1 - \|x\|^2}$ is a strictly convex weak* lower semicontinuous function on (B_Z, w^*) that cannot be extended to a convex function on Z .

4. EXPOSED POINTS

Notice that if a strictly convex function is constant on a face of a compact K , then necessarily that face should be an exposed point of K . Having this in mind, Propositions 3.3 and 3.4 can be rewritten. As in the previous section $Z = W^*$ is a dual Banach space endowed with the weak* topology and we understood all the topological notions referred to that topology.

Proposition 4.1. *Let $f: Z \rightarrow \mathbb{R}$ be a strictly convex lower semicontinuous function which is bounded on compact subsets. If $K \subset Z$ is compact convex, then there exists a G_δ dense set of elements of W exposing points of K at which $f|_K$ is continuous.*

Proof. It follows straightforward from Proposition 3.3. \square

In particular, we retrieve the following result, which is usually proved in the frame of Gâteaux Differentiability Spaces [13, Corollary 2.39 and Theorem 6.2].

Corollary 4.2 (Asplund, Larman–Phelps). *Let Z be a strictly convex dual Banach space. Then every convex compact is the closed convex hull of its exposed points.*

Proof of Theorem 1.2. It follows straightforward from Proposition 3.4. \square

Corollary 4.3. *Assume that $K \in \mathcal{SC}(X)$. Then K is the closed convex hull of its exposed points.*

Proof. Thanks to Theorem 1.1 it can be reduced to the previous corollary. \square

Notice that the previous result is far from being a characterization. For instance, consider $X = C([0, \omega_1])^*$ and $K = (B_X, w^*)$. Then X has the Radon–Nikodým Property and thus there exist *strongly exposed points* of K [1, Theorem 3.5.4]. Nevertheless, Talagrand’s argument in [2, Theorem 5.2.(ii)] shows that $K \notin \mathcal{SC}(X, w^*)$. Indeed, the result of Larman and Phelps mentioned above states that Banach spaces for which each weak* compact convex subset has an exposed point are exactly dual spaces of a Gâteaux Differentiability Space.

Remark 4.4. A point x in a subset C of a normed space $(Z, \|\cdot\|)$ is said to be a *farthest point* in C if there exists $y \in Z$ such that $\|y - x\| \geq \sup\{\|y - c\| : c \in C\}$. If $\|\cdot\|$ is strictly convex then every farthest point of C is exposed by a functional in Z^* . In addition, it was shown in [3] that there exists a weak* compact subset of ℓ_1 that has no farthest points, so the existence of exposed points does not imply the existence of farthest points. On the other hand, suppose that Z is a strictly convex dual Banach space, C is a compact subset of Z and x is a farthest point in C with respect to $y \in Z$. Consider the symmetric $\rho(u, v) = \frac{\|u-y\|^2 + \|v-y\|^2}{2} - \|\frac{u+v}{2} - y\|^2$. Then x is a ρ -denting point of C , that is, admits slices with arbitrarily small ρ -diameter. Indeed, if $\delta = \frac{\varepsilon}{1+2\|x-y\|+2\|y\|}$ then every slice of C that does not meet $B(y, \|y - x\| - \delta)$ has ρ -diameter less than ε .

Typically a variational principle provides strong minimum for certain functions after a small perturbation. But in the compact setting, a lower semicontinuous function already attains its minimum. Nevertheless, inspired by Stegall’s variational principle [4, Theorem 11.6], we have obtained the following result.

Proposition 4.5. *Suppose that $K \in \mathcal{SC}(X)$ and let $f : K \rightarrow \mathbb{R}$ be a lower semicontinuous function. Given $\varepsilon > 0$, there exists an affine continuous function w on K with oscillation less than ε such that $f + w$ attains its minimum exactly at one point. Moreover, if X is a dual Banach space then w can be taken from the predual with norm less than ε .*

Proof. By the embedding it is enough to consider the Banach case. Let m be the minimum of f and take $M > 0$ such that $K \subset MB_X$. Consider the compact set

$$H = \{(x, t) : f(x) \leq t \leq m + \varepsilon M\}$$

and take its convex closed envelop A . By Proposition 2.2, $A \in \mathcal{SC}(X \times \mathbb{R})$. The functional on $X \times \mathbb{R}$ given by $(0, 1)$ attains its minimum on A . Proposition 4.1 provides a small perturbation of the form $(w, 1)$, with $\|w\| < \varepsilon$, attaining its minimum on A at one single point (x_0, t_0) . Notice that $t_0 = f(x_0)$ and $f(x_0) + w(x_0) \leq m + \varepsilon M$. If $y \in K$, then either $f(y) \leq m + \varepsilon M$ and $(y, f(y)) \in A$, or $f(y) > m + \varepsilon M \geq f(x_0) + w(x_0)$. \square

5. ORDINAL INDICES

Let K be a convex and compact subset of a locally convex topological vector space and ρ a symmetric on K . We consider the following set derivations:

$$\begin{aligned} [K]_\varepsilon' &= \{x \in K : x \in S \text{ slice of } K \Rightarrow \rho\text{-diam}(S) \geq \varepsilon\}; \\ \langle K \rangle_\varepsilon' &= \{x \in K : x \in U \text{ open} \Rightarrow \rho\text{-diam}(S) \geq \varepsilon\}. \end{aligned}$$

The iterated derived sets are defined as $[K]_\varepsilon^{\alpha+1} = [[K]_\varepsilon^\alpha]_\varepsilon'$, $\langle K \rangle_\varepsilon^{\alpha+1} = \langle \langle K \rangle_\varepsilon^\alpha \rangle_\varepsilon'$ and intersection in case of limit ordinals. If there exists some ordinal such that $[K]_\varepsilon^\alpha = \emptyset$, then we set $Dz_\rho(K, \varepsilon) = \min\{\alpha : [K]_\varepsilon^\alpha = \emptyset\}$. Otherwise, we take $Dz_\rho(K, \varepsilon) = \infty$, which is beyond the ordinals. The ρ -dentability index of K is

defined by $Dz_\rho(K) = \sup_{\varepsilon > 0} Dz(K, \varepsilon)$. The ρ -Szlenk index of K , $Sz_\rho(K)$, is defined the same way. Obviously $Sz_\rho(K) \leq Dz_\rho(K)$. Set derivations with respect to a symmetric were introduced in [7] in order to characterize dual Banach spaces admitting a dual strictly convex norm.

Proposition 5.1. *Let K be a convex compact subset of a locally convex space. Then the following assertions are equivalent:*

- a) $K \in \mathcal{SC}$;
- b) there exists a symmetric ρ on K such that $Dz_\rho(K) \leq \omega$;
- c) there exists a symmetric ρ on K such that $Dz_\rho(K) \leq \omega_1$.

Proof. Let f be a bounded function witnessing that $K \in \mathcal{SC}$ and assume that f takes values in $[0, 1]$. For a fixed $\varepsilon > 0$, take $N > 1/\varepsilon$ and define the closed convex subsets $F_n = \{x \in K : f(x) \leq 1 - n/N\}$ for $n = 0, \dots, N$. Take

$$\rho(x, y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right).$$

We claim that $[K]_\varepsilon' \subset F_1$. Let $x_0 \in K \setminus F_1$. By the Hahn–Banach theorem, there exists a slice S of K such that $x_0 \in S$ and $S \cap F_1 = \emptyset$. If $x, y \in S$, then $\frac{x+y}{2} \in S$ and $\rho(x, y) \leq 1 - (1 - 1/N) = 1/N$. Thus, $\rho\text{-diam}(S) < \varepsilon$ and $x_0 \notin [K]_\varepsilon'$. By iteration, we get that $[K]_\varepsilon^N \subset F_N$ and hence $[K]_\varepsilon^{N+1} = \emptyset$. Therefore, $Dz_\rho(K, \varepsilon) < \omega$ for each $\varepsilon > 0$.

Now suppose that $Dz_\rho(K) \leq \omega_1$. Notice that indeed $Dz_\rho(K) < \omega_1$. By Corollary 2.9, it suffices to show that K has (*) with slices. For each $n \in \mathbb{N}$ and $\alpha < Dz_\rho(K, 1/n)$ consider the family

$$\mathcal{U}_{n,\alpha} = \left\{ S : S \text{ slice of } K, [K]_{1/n}^{\alpha+1} \cap S = \emptyset, \rho\text{-diam}([K]_{1/n}^\alpha \cap S) < 1/n \right\}.$$

Given distinct $x, y \in K$, take n so that $\rho(x, y) > 1/n$ and let α be the least ordinal such that $\{x, y\} \cap [K]_{1/n}^{\alpha+1}$ is at most a singleton. Then it is clear that there is a slice in $\mathcal{U}_{n,\alpha}$ containing either x or y , and no slice in $\mathcal{U}_{n,\alpha}$ contains both points. \square

Remark 5.2. By using deep results of descriptive set theory, Lancien proved in [8] that there exists an universal function $\psi : [0, \omega_1] \rightarrow [0, \omega_1]$ such that $Dz_{\|\cdot\|}(B_{X^*}) \leq \psi(Sz_{\|\cdot\|}(B_{X^*}))$ whenever X is a Banach space such that $Sz_{\|\cdot\|}(B_{X^*}) < \omega_1$. We do not know if a similar statement holds when the norm is replaced by a symmetric.

We shall show that we cannot change symmetric by metric in Proposition 5.1. That would imply that K is a Gruenhage compact, which is a strictly stronger condition than being in \mathcal{SC} [18, Theorem 2.4]. By [19, Lemma 7.1 and Proposition 7.4], a compact space K is Gruenhage if and only if there exist a countable set D , a family of closed sets $\{A_d : d \in D\}$ and families $(\mathcal{U}_d)_{d \in D}$ of open sets such that the family $\{A_d \cap U : U \in \mathcal{U}_d\}$ is pairwise disjoint for each $d \in D$ and the family $\{A_d \cap U : U \in \mathcal{U}_d\}$ separates the points of K .

Proposition 5.3. *Let K be a compact space. Then the following assertions are equivalent:*

- a) K is Gruenhage;
- b) there exists a metric d on K such that $Sz_d(K) \leq \omega$;
- c) there exists a metric d on K such that $Sz_d(K) \leq \omega_1$.

Proof. If K is a Gruenhage compact space, then the same construction used in the proof of [15, Theorem 2.8] provides a metric on K such that $Sz_d(K) \leq \omega$.

Now assume that d is a metric on K with countable Szlenk index. Let $\mathcal{B} = \bigcup_{m \in \mathbb{N}} \mathcal{B}_m$ be a basis of the metric topology such that every \mathcal{B}_m is discrete. Consider the open sets $U_V^{n,\alpha} = \bigcup \{U : U \text{ open}, \langle K \rangle_{2^{-n}}^\alpha \cap U \subset V\}$ and the families $\mathcal{U}_m^{n,\alpha} =$

$\{U_V^{n,\alpha} : V \in \mathcal{B}_m\}$. Then $\{\langle K \rangle_{2^{-n}}^\alpha \cap U : U \in \mathcal{U}_m^{n,\alpha}\}$ is pairwise disjoint for each $n, m \in \mathbb{N}$ and $\alpha < Dz(K, 2^{-n})$. Given distinct $x, y \in K$ take $V \in \mathcal{B}_m$ such that $x \in V$ and $y \notin V$. Fix n such that $B_d(x, 2^{-n+1}) \subset V$. Let α be the least ordinal so that $x \notin \langle K \rangle_{2^{-n}}^{\alpha+1}$. Then there is an open subset U of K such that $x \in \langle K \rangle_{2^{-n}}^\alpha \cap U$ and $\text{diam}(\langle K \rangle_{2^{-n}}^\alpha \cap U) \leq 2^{-n}$. Thus $x \in \langle K \rangle_{2^{-n}}^\alpha \cap U_V^{n,\alpha} \subset V$, so $y \notin \langle K \rangle_{2^{-n}}^\alpha \cap U_V^{n,\alpha}$. \square

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